9/1/25 Lecture 4

e.g. $2x_1 - x_2 + 2x_3 = 0$
$x_1 + 7x_2 = 2$

Augmented Matrix: $\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 1 & 7 & 0 & 2 \end{array}\right]$

$\vec{b} \in \mathbb{R}^2$
($\vec{b}$ is a vector in $\mathbb{R}^2$)

Coefficient Matrix A $= \begin{bmatrix} 2 & -1 & 2 \\ 1 & 7 & 0 \end{bmatrix}$

Augmented Matrix: $\left[\, A \mid \vec{b} \,\right]$

Vector Equation:

$$\underbrace{x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 7 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Def <u>linear combination</u> of $\vec{v}_1, \ldots \vec{v}_n \in \mathbb{R}^m$ is a vector of the form

$c_1\vec{v}_1 + \ldots c_n\vec{v}_n \in \mathbb{R}^m$ where $c_1, \ldots c_n$ are scalars

In e.g. is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$?

Check:

$$1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 7 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Answer: NO

Now some stuff with rows to rewrite system

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2$$

Def - Let A be an $m \times n$ matrix with columns $\vec{v}_1, \ldots \vec{v}_n \in \mathbb{R}^m$ and $\vec{x} \in \mathbb{R}^n$ be a vector. Then $A\vec{x} = x_1 \vec{v}_1 + \ldots x_n \vec{v}_n$

e.g.
$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 7 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 7 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 7 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} (2,-1,2) \cdot (1,0,-1) \\ (1,7,0) \cdot (1,0,-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Another example of matrix times vector

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 3 & -1 & 0 & 2 \end{bmatrix} \overset{\vec{v}}{\begin{bmatrix} 6 \\ 5 \\ 6 \\ 5 \end{bmatrix}} = \begin{bmatrix} R_1 \cdot \vec{v} \\ R_2 \cdot \vec{v} \\ R_3 \cdot \vec{v} \end{bmatrix} = \begin{bmatrix} 34 \\ 90 \\ 23 \end{bmatrix}$$

Can Rewrite system as:

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 7 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Ex:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$$

Solution:

$$x_1 + 3x_2 + x_4 = 4$$

$$x_1 = 4 - 3x_2 - x_4$$

$$x_2 = x_2$$

$$x_3 = 6 + x_4$$

$$x_4 = x_4$$

$$\vec{x} = \begin{pmatrix} 4 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

ANY column without a PIVOT is a free variable.

Prop: Suppose A & B are $m \times n$ matrices, $\vec{x}, \vec{y} \in \mathbb{R}^n$ are vectors, and $c \in \mathbb{R}$ is a scaler. Then:

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
2. $(A+B)\vec{x} = A\vec{x} + B\vec{x}$
3. $c(A \cdot \vec{x}) = (cA) \cdot \vec{x} = A \cdot (c\vec{x})$